

Quantum Lower Bounds by Entropy Numbers

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Abstract

We use entropy numbers in combination with the polynomial method to derive a new general lower bound for the n -th minimal error in the quantum setting of information-based complexity. As an application, we improve some lower bounds on quantum approximation of embeddings between finite dimensional L_p spaces and of Sobolev embeddings.

1 Introduction

There is one major technique for proving lower bounds in the quantum setting of IBC (information-based complexity) as introduced in [5]. It uses the polynomial method [1] together with a result on approximation by polynomials from [14]. This method has been applied in [5],[9],[19]. Other papers on the quantum complexity of continuous problems use this implicitly by reducing mean computation to the problem in consideration and then using the lower bound for mean computation of [14] directly ([15],[18],[11], [16]).

This approach, however, does not work for the case of approximation of embedding operators in spaces with norms different from the infinity norm. To settle such situations, a more sophisticated way of reduction to known bounds was developed in [6], based on a multiplicativity property of the n -th minimal quantum error.

In this paper we introduce an approach which is new for the IBC quantum setting. We again use the polynomial method of [1], but combine it with methods related to entropy [4]. We derive lower bounds for the n -th minimal quantum error in terms of certain entropy numbers. Similar ideas have been applied before in [17], the model and methods however being different,

see also related work [13]. As an application, we improve the lower bounds [6, 7] on approximation as well as those of [8] by removing the logarithmic factors.

Let us also mention that a modification of the polynomial method based on trigonometric polynomials was used in [2, 3] for proving lower bounds for a type of query different from that introduced in [5], the so-called power-query [16]. Our method can also be applied in this setting and simplifies the analysis from [2, 3]. We comment on this at the end of the paper.

2 Lower bounds by entropy

We work in the quantum setting of IBC as introduced in [5]. We refer to this paper for the needed notions. Let D and K be nonempty sets, let $\mathcal{F}(D, K)$ denote the set of all functions on D with values in K , let $F \subseteq \mathcal{F}(D, K)$ be nonempty, and let G be a normed linear space. Let A be a quantum algorithm from F to G . For any subset $C \subseteq G$ define the function $p_C : F \rightarrow \mathbb{R}$ by

$$p_C(f) = \mathbb{P}\{A(f) \in C\} \quad (f \in F)$$

– the probability that the output of algorithm A at input f belongs to C . This quantity is well-defined for all subsets C since the output of A takes only finitely many values, see [5]. Furthermore, define

$$\mathcal{P}_{A,F} = \text{span}\{p_C : C \subseteq G\} \subseteq \mathcal{F}(F, \mathbb{R})$$

to be the linear span of the functions p_C .

We need some notions related to entropy. We refer to [4] for the definitions. For a nonempty subset W of a normed space G and $k \in \mathbb{N}$ (we use the notation $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$) define the k -th inner entropy number as

$$\begin{aligned} \varphi_k(W, G) &= \sup\{\varepsilon : \text{there exist } u_1, \dots, u_{k+1} \in W \text{ such that} \\ &\quad \|u_i - u_j\| \geq 2\varepsilon \text{ for all } 1 \leq i \neq j \leq k+1\}. \end{aligned} \quad (1)$$

It is worth while mentioning a related notion. The k -th entropy number is defined to be

$$\begin{aligned} \varepsilon_k(W, G) &= \inf\{\varepsilon : \text{there exist } g_1, \dots, g_k \in G \text{ such that} \\ &\quad \min_{1 \leq i \leq k} \|g - g_i\|_G \leq \varepsilon \text{ for all } g \in W\}. \end{aligned} \quad (2)$$

Then

$$\varphi_k(W, G) \leq \varepsilon_k(W, G) \leq 2\varphi_k(W, G), \quad (3)$$

see [4], relations (1.1.3) and (1.1.4). Also observe that the first numbers of both types are related to the radius and diameter of W as follows:

$$\varphi_1(W, G) = \frac{1}{2} \text{diam}(W, G), \quad \varepsilon_1(W, G) = \text{rad}(W, G). \quad (4)$$

Entropy numbers of bounded linear operators (that means, the entropy numbers of the image of the unit ball under the action of the operator) as well as their relation to various s -numbers and to eigenvalues are well-studied, see again [4] and references therein.

Let S be a mapping from F to G and let $e(S, A, F)$ denote the error of quantum algorithm A on F . Our basic lemma relates this error to the dimension of $\mathcal{P}_{A,F}$ and the entropy of $S(F) \subseteq G$.

Lemma 1. (i) Let $k \in \mathbb{N}$ be such that

$$k + 1 > (\log_2 5) \dim \mathcal{P}_{A,F}. \quad (5)$$

Then

$$e(S, A, F) \geq \varphi_k(S(F), G). \quad (6)$$

(ii) If A is an algorithm without queries, then

$$e(S, A, F) \geq \varphi_1(S(F), G). \quad (7)$$

Proof. The first part of the proof is the same for both cases. For case (i) we assume that k satisfies (5), while in case (ii) we set $k = 1$. Let $f_1, \dots, f_{k+1} \in F$ be arbitrary elements and put

$$\varepsilon = \min \{ \|S(f_i) - S(f_j)\| : 1 \leq i \neq j \leq k+1 \}. \quad (8)$$

It suffices to show that

$$e(S, A, F) \geq \varepsilon/2. \quad (9)$$

For $\varepsilon = 0$ this is trivial, so we suppose $\varepsilon > 0$. We assume the contrary of (9), that is

$$e(S, A, F) < \varepsilon/2. \quad (10)$$

By (8), the subsets $V_i \subset G$ defined by

$$V_i = \left\{ g \in G : \|S(f_i) - g\| < \frac{\varepsilon}{2} \right\} \quad (i = 1, \dots, k+1) \quad (11)$$

are disjoint. It follows from (10) and (11) that for $i = 1, \dots, k+1$

$$\mathbb{P}\{A(f_i) \in V_i\} \geq \frac{3}{4}. \quad (12)$$

Let us first complete the proof of (ii): If A has no queries, its output does not depend on $f \in F$, and in particular, the distribution of the random variables $A(f_1)$ and $A(f_2)$ is the same. But then (12) implies

$$\mathbb{P}\{A(f_1) \in V_1 \cap V_2\} \geq 1/2,$$

thus $V_1 \cap V_2 \neq \emptyset$, a contradiction, which proves (9) in case (ii).

Now we deal with case (i). Let \mathcal{C} be the set of all $C \subset G$ of the form

$$C = \bigcup_{i \in I} V_i,$$

with I being any subset of $\{1, \dots, k+1\}$. Clearly,

$$|\mathcal{C}| = 2^{k+1}. \quad (13)$$

Let $\mathcal{P}_{A,F}$ be endowed with the supremum norm

$$\|p\|_\infty = \sup_{f \in F} |p(f)|.$$

We have

$$\|p_C\|_\infty \leq 1 \quad (C \in \mathcal{C}). \quad (14)$$

Moreover,

$$\|p_{C_1} - p_{C_2}\|_\infty \geq \frac{1}{2} \quad (C_1 \neq C_2 \in \mathcal{C}). \quad (15)$$

Indeed, for $C_1 \neq C_2 \in \mathcal{C}$ there is an i with $1 \leq i \leq k+1$ such that $V_i \subseteq C_1 \setminus C_2$ or $V_i \subseteq C_2 \setminus C_1$. Without loss of generality we assume the first. Then, because of (12), we have

$$p_{C_1}(f_i) = \mathbb{P}\{A(f_i) \in C_1\} \geq \mathbb{P}\{A(f_i) \in V_i\} \geq \frac{3}{4},$$

while

$$p_{C_2}(f_i) = \mathbb{P}\{A(f_i) \in C_2\} \leq \mathbb{P}\{A(f_i) \in G \setminus V_i\} \leq \frac{1}{4},$$

hence

$$|p_{C_1}(f_i) - p_{C_2}(f_i)| \geq \frac{1}{2},$$

implying (15). For $p \in \mathcal{P}_{A,F}$ let $B(p, r)$ be the closed ball of radius r around p in $\mathcal{P}_{A,F}$. By (15) the balls $B(p_C, 1/4)$ have disjoint interior for $C \in \mathcal{C}$. Moreover, by (14),

$$\bigcup_{C \in \mathcal{C}} B(p_C, 1/4) \subseteq B(0, 5/4).$$

A volume comparison gives

$$2^{k+1} = |\mathcal{C}| \leq 5^{\dim \mathcal{P}_{A,F}},$$

hence, taking logarithms, we get a contradiction to (5), which completes the proof. \square

Let $e_n^q(S, F)$ denote the n -th minimal quantum error, that is, the infimum of $e(S, A, F)$ taken over all quantum algorithms A from F to G with at most n queries (see [5]). As an immediate consequence of Lemma 1, and also for later use, we note the following.

Corollary 1.

$$\frac{1}{2} \text{diam}(S(F), G) \leq e_0^q(S, F) \leq \text{rad}(S(F), G). \quad (16)$$

Proof. The lower bound follows from Lemma 1 (ii) and (4). The upper bound is obtained by taking for any $\delta > 0$ a point $g_\delta \in G$ with

$$\|S(f) - g_\delta\| \leq \text{rad}(S(F), G) + \delta \quad \text{for all } f \in F$$

and then using the trivial algorithm which outputs g_δ for all $f \in F$, with probability 1. \square

Next we recall some facts from [5], section 4. Let $L \in \mathbb{N}$ and let to each $u = (u_1, \dots, u_L) \in \{0, 1\}^L$ an $f_u \in \mathcal{F}(D, K)$ be assigned such that the following is satisfied:

Condition (I): For each $t \in D$ there is an ℓ , $1 \leq \ell \leq L$, such that $f_u(t)$ depends only on u_ℓ , in other words, for $u, u' \in \{0, 1\}^L$, $u_\ell = u'_\ell$ implies $f_u(t) = f_{u'}(t)$.

The following result was shown in [5], Corollary 2, based on the idea of the quantum polynomial method [1].

Lemma 2. *Let $L \in \mathbb{N}$ and assume that $(f_u)_{u \in \{0,1\}^L} \subseteq \mathcal{F}(D, K)$ satisfies condition (I). Let $n \in \mathbb{N}_0$ and let A be a quantum algorithm from $\mathcal{F}(D, K)$ to G with n quantum queries. Then for each subset $C \subseteq G$,*

$$p_C(f_u) = p_C(f_{(u_1, \dots, u_L)}),$$

considered as a function of the variables $u_1, \dots, u_L \in \{0, 1\}$, is a real multilinear polynomial of degree at most $2n$.

Now we are ready to state the new lower bound on the n -th minimal quantum error.

Proposition 1. *Let D, K be nonempty sets, let $F \subseteq \mathcal{F}(D, K)$ be a nonempty set of functions, G a normed space, $S : F \rightarrow G$ a mapping, and $L \in \mathbb{N}$. Suppose $\mathcal{L} = (f_u)_{u \in \{0,1\}^L} \subseteq \mathcal{F}(D, K)$ is a system of functions satisfying condition (I). Then*

$$e_n^q(S, F) \geq \varphi_k(S(F \cap \mathcal{L}), G)$$

whenever $k, n \in \mathbb{N}$ satisfy $2n \leq L$ and

$$k + 1 > (\log_2 5) \left(\frac{eL}{2n} \right)^{2n}. \quad (17)$$

Proof. Let $n \in \mathbb{N}$ with $2n \leq L$ and let A be a quantum algorithm from F to G with no more than n queries. Note that, by definition, a quantum algorithm from $F \subseteq \mathcal{F}(D, K)$ to G is always also a quantum algorithm from $\mathcal{F}(D, K)$ to G (see [5], p. 7). We show that

$$e(S, A, F) \geq \varphi_k(S(F \cap \mathcal{L}), G) \quad (18)$$

for all $k \in \mathbb{N}$ satisfying (17). Let $\mathcal{M}_{L,2n}$ be the linear space of real multilinear polynomials in L variables of degree not exceeding $2n$. Since $2n \leq L$, its dimension is

$$\dim \mathcal{M}_{L,2n} = \sum_{i=0}^{2n} \binom{L}{i} \leq \left(\frac{eL}{2n} \right)^{2n} \quad (19)$$

(see, e.g., [12], (4.7) on p. 122, for the inequality). Set

$$U = \{u \in \{0,1\}^L : f_u \in F\}$$

and let $\mathcal{M}_{L,2n}(U)$ denote the space of all restrictions of functions from $\mathcal{M}_{L,2n}$ to U . Clearly,

$$\dim \mathcal{M}_{L,2n}(U) \leq \dim \mathcal{M}_{L,2n}. \quad (20)$$

Define

$$\Psi : \mathcal{P}_{A,F \cap \mathcal{L}} \rightarrow \mathcal{F}(U, \mathbb{R})$$

by setting for $p \in \mathcal{P}_{A,F \cap \mathcal{L}}$ and $u \in U$

$$(\Psi p)(u) = p(f_u).$$

Obviously, Ψ is linear, moreover, for $C \subseteq G$

$$(\Psi p_C)(u) = p_C(f_u) \quad (u \in U).$$

By Lemma 2, $p_C(f_u)$, as a function of $u \in U$, is the restriction of an element of $\mathcal{M}_{L,2n}$ to U . Hence,

$$\Psi p_C \in \mathcal{M}_{L,2n}(U),$$

and by linearity and the definition of $\mathcal{P}_{A,F \cap \mathcal{L}}$ as the linear span of functions p_C , we get

$$\Psi(\mathcal{P}_{A,F \cap \mathcal{L}}) \subseteq \mathcal{M}_{L,2n}(U).$$

Furthermore, Ψ is one-to-one, since $\{f_u : u \in U\} = F \cap \mathcal{L}$. Using (19) and (20) it follows that

$$\dim \mathcal{P}_{A,F \cap \mathcal{L}} \leq \dim \mathcal{M}_{L,2n}(U) \leq \left(\frac{eL}{2n}\right)^{2n}.$$

Consequently, for k satisfying (17),

$$k + 1 > (\log_2 5) \dim \mathcal{P}_{A,F \cap \mathcal{L}}.$$

Now (18) follows from Lemma 1. \square

3 Some applications

For $N \in \mathbb{N}$ and $1 \leq p \leq \infty$, let L_p^N denote the space of all functions $f : \{1, \dots, N\} \rightarrow \mathbb{R}$, equipped with the norm

$$\|f\|_{L_p^N} = \left(\frac{1}{N} \sum_{i=1}^N |f(i)|^p\right)^{1/p}$$

if $p < \infty$,

$$\|f\|_{L_\infty^N} = \max_{1 \leq i \leq N} |f(i)|,$$

and let $\mathcal{B}(L_p^N)$ be its unit ball. Define $J_{pq}^N : L_p^N \rightarrow L_q^N$ to be the identity operator $J_{pq}^N f = f$ ($f \in L_p^N$).

As already mentioned, the lower bound for approximation of J_{pq}^N was obtained using a multiplicativity property of the n -th minimal quantum error ([6], Proposition 1). The result involved some logarithmic factors of negative power ([6], Proposition 6). Based on Proposition 1 above we improve this bound by removing the logarithmic factors.

Proposition 2. *Let $1 \leq p, q \leq \infty$. There is a constant $c > 0$ such that for all $n \in \mathbb{N}_0$, $N \in \mathbb{N}$ with $n \leq cN$*

$$e_n^q(J_{pq}^N, \mathcal{B}(L_p^N)) \geq \frac{1}{8}.$$

Proof. It suffices to prove the case $p = \infty$, $q = 1$. We put $L = N$ and $f_u = u$ for $u \in \{0, 1\}^N$. Clearly, the system $\mathcal{L} = (f_u)_{u \in \{0, 1\}^N}$ satisfies condition (I) and

$$\mathcal{L} \subset \mathcal{B}(L_\infty^N). \quad (21)$$

Let $\{f_{u_i} : 1 \leq i \leq k+1\}$ be a maximal system with

$$\|f_{u_i} - f_{u_j}\|_{L_1^N} \geq \frac{1}{4} \quad (1 \leq i \neq j \leq k+1). \quad (22)$$

Maximality implies

$$\{0, 1\}^N = \bigcup_{i=1}^{k+1} \left\{ u \in \{0, 1\}^N : \|f_u - f_{u_i}\|_{L_1^N} < \frac{1}{4} \right\}.$$

On the other hand,

$$\begin{aligned} 2^N &\leq \sum_{i=1}^{k+1} \left| \left\{ u \in \{0, 1\}^N : \|f_u - f_{u_i}\|_{L_1^N} < \frac{1}{4} \right\} \right| \\ &\leq (k+1) \sum_{0 \leq j < N/4} \binom{N}{j} \leq (4e)^{N/4}, \end{aligned}$$

again by [12], (4.7) on p. 122. It follows that

$$k+1 \geq 2^N (4e)^{-N/4} = 2^{c_1 N}, \quad (23)$$

with $c_1 = \frac{1}{4} \log_2(\frac{4}{e}) > 0$, hence $k \in \mathbb{N}$. From (22) we obtain

$$\varphi_k(J_{\infty, 1}^N(\mathcal{L}), L_1^N) \geq \frac{1}{8}. \quad (24)$$

Consider the function $g : (0, 1] \rightarrow \mathbb{R}$,

$$g(x) = x \left(\log_2 e + \log_2 \frac{1}{x} \right).$$

It is elementary to check that g is monotonely increasing. Moreover $g(x) \rightarrow 0$ as $x \rightarrow 0$. Choose $0 < c_2 \leq 1$ in such a way that

$$g(x) < \frac{c_1}{2} \quad (0 < x \leq c_2). \quad (25)$$

Now put

$$c = \min \left(\frac{c_1}{2 \log_2 \log_2 5}, \frac{c_2}{2}, \frac{1}{2} \right) \quad (26)$$

and assume

$$n \leq cN. \quad (27)$$

If $n = 0$, Corollary 1 gives

$$e_0^q(J_{\infty,1}^N, \mathcal{B}(L_\infty^N)) = \|J_{\infty,1}^N\| = 1. \quad (28)$$

Hence we can suppose that $n \geq 1$, which, by (27), implies $N \geq c^{-1}$. Consequently, from (26),

$$\frac{\log_2 \log_2 5}{N} \leq \frac{c_1}{2}. \quad (29)$$

Since by (26) and (27), $2n/N \leq 2c \leq c_2$, we get from (25)

$$\frac{2n}{N} \left(\log_2 e + \log_2 \frac{N}{2n} \right) < \frac{c_1}{2}, \quad (30)$$

and therefore, with (29),

$$\frac{\log_2 \log_2 5}{N} + \frac{2n}{N} \left(\log_2 e + \log_2 \frac{N}{2n} \right) < c_1. \quad (31)$$

This implies, using also (23),

$$(\log_2 5) \left(\frac{eN}{2n} \right)^{2n} < 2^{c_1 N} \leq k + 1. \quad (32)$$

Since we have $k, n \in \mathbb{N}$ satisfying (32), and moreover, by (26) and (27), $2n \leq N$, we can use Proposition 1 together with (21) and (24) to conclude

$$e_n^q(J_{\infty,1}^N, \mathcal{B}(L_\infty^N)) \geq \varphi_k(J_{\infty,1}^N(\mathcal{L}), L_1^N) \geq \frac{1}{8}.$$

□

Using Proposition 2 we can also remove the logarithmic factors in another lower bound – for Sobolev embeddings $J_{pq} : W_p^r([0, 1]^d) \rightarrow L_q([0, 1]^d)$, see [7] for the notation and Proposition 2 of that paper for the previous result. The following can be derived from Proposition 2 using the same argument as in [7], p. 43, relations (87) and (88).

Corollary 2. *Let $1 \leq p, q \leq \infty$, $r, d \in \mathbb{N}$, and assume $\frac{r}{d} > \max\left(\frac{1}{p}, \frac{2}{p} - \frac{2}{q}\right)$. Then there is a constant $c > 0$ such that for all $n \in \mathbb{N}$*

$$e_n^q(J_{pq}, \mathcal{B}(W_p^r([0, 1]^d))) \geq cn^{-r/d}.$$

Furthermore, the lower bounds from [6] were also used in [8], Proposition 3 and Corollary 3. Using Proposition 2, these results can be improved in the respective way, too. We omit the details.

Let us finally comment on lower bounds for power queries introduced in [16]. An inspection of the proof of Lemma 1 shows that the type of query is not used at all in the argument, so the statement also holds for power queries. One part of the argument in both [2, 3] consists of proving that for a quantum algorithm with at most n power queries and for a suitable subset $F_0 \subseteq F$, which can be identified with the interval $[0, 1]$, the respective space \mathcal{P}_{A, F_0} is contained in the (complex) linear span of functions $e^{2\pi i \alpha t}$ ($t \in [0, 1]$), with frequencies α from a set of cardinality not greater than c^n for some $c > 0$, hence, $\dim \mathcal{P}_{A, F_0} \leq 2c^n$. Moreover, also $S(F_0)$ can be identified with the unit interval. Now Lemma 1 above directly yields the logarithmic lower bounds from [2, 3], since the k -th inner entropy number of the unit interval is k^{-1} .

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